Chapter 20

Markov and Semi-Markov Reward Processes and Stochastic Annuities

20.1. Reward processes

The association of a sum of money a state of the system and a state transition assumes great relevance in the study of financial events. This can be done by linking a reward structure to a stochastic process. This structure can be thought of as a random variable associated with the state occupancies and transitions (see Howard (1971)).

The rewards can be of different kinds, but in the financial environment only amounts of money will be considered as rewards. These amounts can be positive if they are seen as a benefit for the system, and negative if they are considered as a cost.

In this chapter, reward structures for discrete-time Markov and semi-Markov processes and how they can be considered a generalization of deterministic annuities will be described. Only the case of discrete-time reward structures and their relations to the discrete-time annuities will be presented.

A simple classification scheme of the different kinds of Discrete-time Markov ReWard Processes (DTMRWP) and Semi-Markov ReWard Processes (DTSMRWP) given in Janssen and Manca (2006, 2007) will be reported.

Process classification

<table>
<tr><td colspan="3">Homogenous</td></tr>
<tr><td colspan="3">Non-homogenous</td></tr>
<tr><td colspan="3"></td></tr>
<tr><td colspan="3">Continuous time</td></tr>
<tr><td colspan="3">Discrete-time</td></tr>
<tr><td colspan="3"></td></tr>
<tr><td colspan="3">Not discounted</td></tr>
<tr><td rowspan="3">Discounted</td><td colspan="2">Fixed interest rate</td></tr>
<tr><td rowspan="2">Variable interest rate</td><td>Homogenous interest law</td></tr>
<tr><td>Non-homogenous interest law</td></tr>
</table>

Reward classification

<table>
<tr><td colspan="2">Time fixed rewards</td></tr>
<tr><td rowspan="2">Time variable rewards</td><td>Homogenous rewards</td></tr>
<tr><td>Non-homogenous rewards</td></tr>
<tr><td colspan="2"></td></tr>
<tr><td colspan="2">Transition (impulse) rewards</td></tr>
<tr><td rowspan="5">Permanence (rate) rewards</td><td>Immediate</td></tr>
<tr><td>Due</td></tr>
<tr><td></td></tr>
<tr><td>Independent on next transition</td></tr>
<tr><td>Dependent on next transition</td></tr>
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Some clarifications as regards the homogenity concept are necessary.

It is assumed that a phenomenon depends on time. We follow the phenomenon in the interval times $\left[s_1, t_1\right]$ and $\left[s_2, t_2\right]$ where $t_1 - s_1 = t_2 - s_2$. If the phenomenon behaves in the same way in the two time intervals and in each interval for the same time period, we say that it is homogenous. On the other hand, in the case in which the phenomenon changes not only for time duration but also because of the initial time, then the phenomenon is non-homogenous.

In general, this distinction is made in the stochastic processes environment, but also, as described in previous chapters, an interest rate law can be homogenous or non-homogenous. It is homogenous if the discount factor is a function of only the length of the financial operation, and is non-homogenous if the discount factor also takes into account not only the duration but also the initial time of the operation.

For the same reason, rewards can also be fixed in time, can depend only on the duration or can be non-homogenous in time.

It should be stated that in finance and insurance problems reward processes without discount do not normally make sense, but in some reliability problems they could have some meaning. Furthermore, the absence of interest rates simplifies the model. In this chapter, we will develop only discounted processes, the non-discounted DTMRWP and DTSMRWP description can be found in Janssen and Manca (2007). A very short description of reward processes with the study of some properties can be found in Rolski *et al.* (1999).

In a discrete-time process and as a first approach, the rewards that depend on permanence in the state could be considered as a generalization of discrete-time annuity. As for the annuities, there are *immediate permanence* rewards that are paid at the end of a period and *due permanence* rewards that are paid at beginning of a period.

All the hypotheses imply different formulae of the system evolution equation. The general relations in both homogenous and non-homogenous environments will be given.

Discounting factors

As regards the financial notations, it is assumed that we are working in a general environment with variable interest rates. In the homogenous case, the following

$$r(1), r(2), \ldots, r(t), \ldots$$

will denote the interest rates and

$$\nu(t) = \begin{cases} 1 & \text{if} \quad t = 0, \\ \prod_{h=1}^{t} \left(1 + r(h)\right)^{-1} & \text{if} \quad t > 0, \end{cases} \tag{20.1}$$

the *t*-period discount factor, if it begins at time 0. In this case, we can also obtain:

$$\nu(s,t) = \begin{cases} 1 & \text{if} \quad t = s, \\ \prod_{h=s+1}^{t} \left(1 + r(h)\right)^{-1} & \text{if} \quad t > s. \end{cases} \tag{20.2}$$

In the non-homogenous interest rate case, the following notations will be used:

$$r(s, s+1), r(s, s+2), \ldots, r(s, s+t), \ldots,$$

for the discrete-time non-homogenous interest rates and:

$$\dot{v}(s,t)=\begin{cases}1 & \text{if} \quad t=s,\\ \prod_{h=s+1}^{t}\left(1+r(s,h)\right)^{-1} & \text{if} \quad t>s,\end{cases} \tag{20.3}$$

for the non-homogenous discount factors.

Reward notation

(i) $\psi_i,\ \psi_i(t),\ \psi_i(s,t)$ denote the reward that is given for permanence in the *i*th state; it is also called rate reward (see Qureshi and Sanders (1994)); the first is paid in cases in which the period amount in state *i* is constant in time, the second when the payment is a function of the state and of the duration inside the state (homogenous payment) and the third when there is a non-homogenous period amount (the payment is a function of the state, the time of entrance into the state and the time of payment). $\boldsymbol{\psi}$ represents the vector of these rewards.

(ii) $\psi_{ij},\psi_{ij}(t),\psi_{ij}(s,t)$ have the same meaning as given previously, the difference being that, in this case, the rewards depend on the future transition. $\boldsymbol{\Psi}$ represents the related matrix. It should be said that these kinds of permanence rewards are usually presented in the other works (see Papadopoulou and Tsaklidis (2006)) and can be seen as a generalization of case (i). In a financial environment, this kind of generalization will not make sense, so we will not present them; the interested reader can refer to Janssen and Manca (2006) and (2007).

(iii) $\gamma_{ij},\ \gamma_{ij}(t),\ \gamma_{ij}(s,t)$ denote the reward that is given for the transition from the *i*th state to the *j*th one (impulse reward); the difference between the three symbols is the same as in the previous cases. $\boldsymbol{\Gamma}$ is the matrix of the transition rewards.

The different kinds of ψ rewards represent an annuity that is paid due to remaining in a state. In the *immediate case*, the reward will be paid at the end of the period before the transition; in the *due case* the reward will be paid at the beginning of the period. On the other hand, γ represents lump sums that, theoretically, are paid at the instant of transition.

As far as the impulse reward γ is concerned, in the case of discounting it is only necessary to calculate the present value of the lump sum paid at the moment of the related transition and that does not present any difficulties.

Reward structure can be considered a very general structure linked to the problem being studied. The reward random variable evolves together with the evolution of the Markov or semi-Markov process with which it is linked. When the considered system, which evolves dynamically in a random way, is in a state, then a reward of type ψ can be paid; once there is a transition, an impulse reward of γ type can be paid.

This behavior is particularly efficient at constructing models which are useful for following, for example, the dynamic evolution of insurance problems.

Usually, in fact, permanence in a state involves the periodic payment of a premium or the periodic receipt of a claim. Furthermore, the transition from one state to another can often give rise to some other cost or benefit.

In the last part of this section, some matrix operation notation useful for describing the evolution equation of the reward processes will be given.

Matrix operations

Given the two matrices **A**, **B** with the notations

$$\mathbf{A} * \mathbf{B} \text{ and } \mathbf{A} \cdot \mathbf{B}$$

respectively the usual row column and the element by element matrix multiplication are denoted. It is clear that in the first case the number of columns in **A** should be equal to the number of the rows in **B** and that in the second operation the two matrices should have the same order of rows and columns.

Definition 20.1 *Given two matrices* $\mathbf{A},\mathbf{B}$ *that have row order equal to m and column order equal to n, the following operation is defined:*

$$\mathbf{c} = \mathbf{A} \circ \mathbf{B} \tag{20.4}$$

where **c** *is the m elements vector in which the ith component is obtained in the following way:*

$$c(i) = \sum_{j=1}^{n} a_{ij} b_{ij} = \mathbf{a}_{i*} * \mathbf{b}_{i*}. \tag{20.5}$$

20.2. Homogenous and non-homogenous DTMRWP

In our opinion, Markov reward processes should be considered a class of stochastic processes, each having different evolution equations. The differences from the analytic point of view can be considered irrelevant but from the algorithmic point of view the differences are very significant and in the construction of the algorithm the differences must be taken into account.

V_i and $\ddot{V}_i$ represent the *mean present value* of the *rewards* (RMPV) paid in the investigated horizon time in the homogenous immediate and due cases respectively.

For the sake of classification, first we present the simplest evolution equation case in immediate and due hypotheses and only in the homogenous case; subsequently, only the general relations in the discrete-time environment will be given.

The immediate homogenous Markov evolution equation in the case of fixed permanence and without transition rewards is the first relation presented. The DTMRWP present value after one payment is:

$$V_i(1) = (1+r)^{-1}\psi_i = (1+r)^{-1}\psi_i, \tag{20.6}$$

after two payments,

$$V_i(2) = (1+r)^{-1}\psi_i + \nu^2 \sum_{k=1}^{m} p_{ik}^{(1)}\psi_k = V_i(1) + \nu^2 \sum_{k=1}^{m} p_{ik}^{(1)}\psi_k, \tag{20.7}$$

and in general, taking into account the recursive nature of relations, at the n^{th} period is:

$$V_i(n) = V_i(n-1) + \nu^n \sum_{k=1}^{m} p_{ik}^{(n-1)}\psi_k, \tag{20.8}$$

that in matrix form becomes:

$$\mathbf{V}(n) = \nu\boldsymbol{\psi} + \cdots + \left(\nu^n \mathbf{P}^{(n-1)}\right) * \boldsymbol{\psi} \tag{20.9}$$

Now the related due case is given:

$$\ddot{V}_i(1) = \psi_i,$$

$$\ddot{V}_i(2) = \psi_i + (1+r)^{-1} \sum_{k=1}^{m} p_{ik}\psi_k = \ddot{V}_i(1) + (1+r)^{-1} \sum_{k=1}^{m} p_{ik}^{(1)}\psi_k, \tag{20.10}$$

$$\ddot{V}_i(n) = \ddot{V}_i(n-1) + (1+r)^{-n+1} \sum_{k=1}^{m} p_{ik}^{(n-1)}\psi_k, \tag{20.11}$$

that in matrix form is:

$$\ddot{\mathbf{V}}(n) = \mathbf{I} * \boldsymbol{\psi} + \left(\nu\mathbf{P}\right) * \boldsymbol{\psi} + \cdots + \left(\nu^{n-1}\mathbf{P}^{(n-1)}\right) * \boldsymbol{\psi} \tag{20.12}$$

Now the general case with variable permanence, transition rewards and interest rates is presented. The present value after one period is:

$$V_i(1) = \nu(1)\left(\psi_i(1) + \sum_{j=1}^{m} p_{ij}\gamma_{ij}(1) \right), \tag{20.13}$$

after two payments,

$$\begin{aligned} V_i(2) &= \nu(1)\left(\psi_i(1) + \sum_{j=1}^{m} p_{ij}\gamma_{ij}(1) \right) \\ &\quad + \nu(2)\sum_{k=1}^{m} p_{ik}\left(\psi_k(2) + \sum_{j=1}^{m} p_{kj}\gamma_{kj}(2) \right) \\ &= V_i(1) + \nu(2)\sum_{k=1}^{m} p_{ik}\left(\psi_k(2) + \sum_{j=1}^{m} p_{kj}\gamma_{kj}(2) \right), \end{aligned} \tag{20.14}$$

and in general, taking into account the recursive nature of relations, at the n^{th} period is:

$$V_i(n) = V_i(n-1) + \nu(n)\sum_{k=1}^{m} p_{ik}^{(n-1)}\left(\psi_k(n) + \sum_{j=1}^{m} p_{kj}\gamma_{kj}(n) \right). \tag{20.15}$$

This relation can be written in matrix notation in the following way:

$$\begin{aligned} &\mathbf{V}(n) = \nu(1)\boldsymbol{\psi}(1) + \cdots + \left(\nu(n)\mathbf{P}^{(n-1)} \right) * \boldsymbol{\psi}(n) \\ &+\nu(1)\left(\mathbf{P} \circ \boldsymbol{\Gamma}(1) \right) + \cdots + \left(\nu(n)\mathbf{P}^{(n-1)} \right) * \left(\mathbf{P} \circ \boldsymbol{\Gamma}(n) \right). \end{aligned} \tag{20.16}$$

In the case of one period payment due, i.e. the permanence reward is paid at the beginning of the period and the transition reward at the end, we have:

$$\ddot{V}_i(1) = \psi_i(1) + \nu(1)\sum_{j=1}^{m} p_{ij}\gamma_{ij}(1), \tag{20.17}$$

with two payments we obtain:

$$\begin{aligned} \ddot{V}_i(2) &= \psi_i(1) + \nu(1)\sum_{k=1}^{m} p_{ik}\gamma_{ik}(1) \\ &\quad + \nu(1)\sum_{j=1}^{m} p_{ij}\psi_j(2) + \nu(2)\sum_{k=1}^{m} p_{ik}\sum_{j=1}^{m} p_{kj}\gamma_{kj}(2) \\ &= \ddot{V}_i(1) + \nu(1)\sum_{k=1}^{m} p_{ik}\psi_k(1) + \nu(2)\sum_{k=1}^{m} p_{ik}\sum_{j=1}^{m} p_{kj}\gamma_{kj}(2). \end{aligned} \tag{20.18}$$

At last, the general relation in the due homogenous Markov case is:

$$\ddot{V}_i(n) = \ddot{V}_i(n-1) + \nu(n)\sum_{k=1}^{m} p_{ik}^{(n-1)} \sum_{j=1}^{m} p_{kj}\gamma_{kj}(n) + \nu(n-1)\sum_{k=1}^{m} p_{ik}^{(n-1)}\psi_k(n), \quad (20.19)$$

which in matrix notation is:

$$\begin{aligned}\ddot{\mathbf{V}}(n) &= \mathbf{I} * \boldsymbol{\psi}(1) + \left(\nu(1)\mathbf{P}\right) * \boldsymbol{\psi}(2) + \cdots + \left(\nu(n-1)\mathbf{P}^{(n-1)}\right) * \boldsymbol{\psi}(n) \\ &+ \left(\nu(1)\mathbf{P}\right) \circ \boldsymbol{\Gamma}(1) + \cdots + \left(\nu(n)\mathbf{P}^{(n-1)}\right) * \left(\mathbf{P} \circ \boldsymbol{\Gamma}(n)\right).\end{aligned} \quad (20.20)$$

Now the non-homogenous formulae with non-homogenous interest rates and rewards are reported. The first gives the immediate case, that is:

$$V_i(s,t) = V_i(s,t-1) + \dot{\nu}(s,t)\sum_{k=1}^{m} p_{ik}^{(n-1)}(s)\left(\psi_k(s,t) + \sum_{j=1}^{m} p_{kj}(t)\gamma_{kj}(s,t)\right), \quad (20.21)$$

where $t = s + n$.

In matrix form, (20.21) becomes:

$$\begin{aligned}\mathbf{V}(s,t) &= \dot{\nu}(s,s+1)\boldsymbol{\psi}(s,s+1) + \left(\dot{\nu}(s,s+1)\mathbf{P}(s+1)\right) \circ \boldsymbol{\Gamma}(s,s+1) \\ &+ \cdots + \left(\dot{\nu}(s,t)\mathbf{P}^{(n-1)}(s)\right) * \boldsymbol{\psi}(s,t) + \left(\dot{\nu}(s,t)\mathbf{P}^{(n-1)}(s)\right) * \left(\mathbf{P}(t) \circ \boldsymbol{\Gamma}(s,t)\right),\end{aligned} \quad (20.22)$$

where $\mathbf{P}^{(n)}(s) = \mathbf{P}(s+1) * \mathbf{P}(s+2) * \cdots * \mathbf{P}(t)$ and $\mathbf{P}(s)$ is the non-homogenous transition matrix at time *s*.

The related due case has the following notation:

$$\begin{aligned}&\ddot{V}_i(s,t) = \ddot{V}_i(s,t-1) + \dot{\nu}(s,t-1)\sum_{k=1}^{m} p_{ik}^{(n-1)}(s)\psi_k(s,t) \\ &+\dot{\nu}(s,t)\sum_{k=1}^{m} p_{ik}^{(n-1)}(s)\sum_{j=1}^{m} p_{kj}(t)\gamma_{kj}(s,t),\end{aligned} \quad (20.23)$$

which in matrix formula becomes:

$$\begin{aligned}\ddot{\mathbf{V}}(s,t) &= \boldsymbol{\psi}(s,s+1) + \left(\dot{\nu}(s,s+1)\mathbf{P}(s)\right) * \boldsymbol{\psi}(s,s+2) + \cdots \\ &+\left(\dot{\nu}(s,t-1)\mathbf{P}^{(n-1)}(s)\right) * \boldsymbol{\psi}(s,t) + \left(\dot{\nu}(s,s+1)\mathbf{P}(s+1)\right) \circ \boldsymbol{\Gamma}(s,s+1) \\ &+\cdots + \left(\dot{\nu}(s,t)\mathbf{P}^{(n-1)}(s)\right) * \left(\mathbf{P}(t) \circ \boldsymbol{\Gamma}(s,t)\right).\end{aligned} \quad (20.24)$$

Remark 20.1 In this section, general formulae were presented. In the construction of the algorithms the differences between the possible cases should be taken into account and it is possible to construct a generalization. For example, in the non-discounting case $\nu(k)=1,\ k=1,\dots,n$ can be stated.

20.3. Homogenous and non-homogenous DTSMRWP

20.3.1. *The immediate cases*

20.3.1.1. *First model*

We assume that all the rewards are discounted at time 0 in the homogenous case and at time *s* in the non-homogenous case. Let us point out that these models, as we will see below, are very important for insurance applications. In the first formulation of this case we suppose that:

a) rewards are fixed in time;

b) rewards are given only for the permanence in the state;

c) rewards are paid at the end of the period;

d) interest rate *r* is fixed.

In this case, $V_i(t)$ represents the mean present value of all the rewards (RMPV) paid or received in a time *t*, given that at time 0 the system is in state *i*.

Under these hypotheses, a similar reasoning as before leads to the following result for the evolution equation, firstly for the homogenous case. Trivially it results in:

$$\begin{aligned} V_i(0) &= 0, \\ V_i(1) &= \left(1-H_i(1)\right)\psi_i \nu^1 + \sum_{k=1}^{m} b_{ik}(1)\psi_i \nu^1 + \sum_{k=1}^{m}\sum_{\vartheta=1}^{1} b_{ik}(\vartheta)V_k(1-\vartheta)\nu^1 \\ &= \psi_i \nu^1, \end{aligned} \tag{20.25}$$

and in general:

$$V_i(t) = (1-H_i(t))\psi_i a_{\overline{t}|r} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\psi_i a_{\overline{\vartheta}|r} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)V_k(t-\vartheta)\nu^{\vartheta}. \tag{20.26}$$

For the non-homogenous case, this last result becomes:

$$\begin{aligned} V_i(s,t) &= (1-H_i(s,t))\psi_i a_{\overline{t-s}|r} + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\psi_i a_{\overline{\vartheta-s}|r} \\ &+ \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu^{\vartheta-s}. \end{aligned} \quad (20.27)$$

To explain these results, we divide the evolution equation into three parts. The meaning is the same as given in the previous cases but we use annuity formulae.

Let us just give the following comments:

– The term $(1-H_i(s,t))\psi_i a_{\overline{t-s}|r}$ represents the present value of the rewards obtained without state changes. More precisely, $(1-H_i(s,t))$ is the probability to remain in state i and $\psi_i a_{\overline{t-s}|r}$ is the present value of a constant annuity of t-s installments ψ_i.

– The term $\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\psi_i a_{\overline{\vartheta-s}|r}$ gives the present value of the rewards obtained before the change of state.

– The term $\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu^{\vartheta-s}$ gives the present value of the rewards paid or earned after the transitions and as the change of state happens at time ϑ, it is necessary to discount the reward values at time s.

As for DTMRWP we will give the matrix equation of each given relation.

To present the matrix form of the previous relations we have to define the following matrices:

$$D_{ij}(t) = \begin{cases} 1-H_i(t) & \text{if} \quad i=j \\ 0 & \text{if} \quad i \neq j \end{cases}, \quad D_{ij}(s,t) = \begin{cases} 1-H_i(s,t) & \text{if} \quad i=j \\ 0 & \text{if} \quad i \neq j \end{cases}.$$

Relations (20.26) and (20.27) respectively become in matrix form:

$$\begin{aligned} \mathbf{V}(t) &= \left(\mathbf{D}(t) * \mathbf{1}\right)\cdot\left(\boldsymbol{\psi} a_{\overline{t}|r}\right) + \sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta) * \mathbf{1}\right)\cdot\left(\boldsymbol{\psi} a_{\overline{\vartheta}|r}\right) \\ &+ \sum_{\vartheta=1}^{t}\mathbf{B}(\vartheta) * \left(\mathbf{V}(t-\vartheta)\nu^{\vartheta}\right), \end{aligned} \quad (20.28)$$

$$\mathbf{V}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1} \right) \cdot \left(\psi a_{\overline{t-s}|r} \right) + \sum_{\vartheta=s+1}^{t} \left(\mathbf{B}(s,\vartheta) * \mathbf{1} \right) \cdot \left(\psi a_{\overline{\vartheta-s}|r} \right)$$
$$+ \sum_{\vartheta=1}^{t} \mathbf{B}(s,\vartheta) * \left(\mathbf{V}(\vartheta,t) \nu^{\vartheta-s} \right),$$

where **1**, as specified in previous chapters, represents the sum vector whose elements are all equal to 1.

20.3.1.2. *Second model*

Now we introduce the case of variable interest rates with as assumptions:

a) rewards are fixed in time;

b) rewards are given only for the permanence in the state;

c) rewards are paid at the end of the period;

d) interest rate r is variable.

Under these hypotheses, it can be shown that we obtain the following formulae:

$$V_i(t) = (1 - H_i(t))\psi_i \sum_{h=1}^{t} \nu(h) + \sum_{k=1}^{m} \sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\psi_i \sum_{h=1}^{\vartheta} \nu(h) + \sum_{k=1}^{m} \sum_{\vartheta=1}^{t} b_{ik}(\vartheta) V_k(t-\vartheta)\nu(\vartheta), \quad (20.29)$$

$$V_i(s,t) = (1 - H_i(s,t))\psi_i \sum_{h=s+1}^{t} \nu(s,h) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\psi_i \sum_{h=s+1}^{\vartheta} \nu(s,h) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) V_k(\vartheta,t)\nu(s,\vartheta). \quad (20.30)$$

The matrix forms related to (20.29) and (20.30) are:

$$\mathbf{V}(t) = \left(\mathbf{D}(t) * \mathbf{1} \right) \cdot \left(\psi \vec{a}(t) \right) + \sum_{\vartheta=1}^{t} \left(\mathbf{B}(\vartheta) * \mathbf{1} \right) \cdot \left(\psi \vec{a}(\vartheta) \right) + \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) * \left(\mathbf{V}(t-\vartheta)\nu(\vartheta) \right), \quad (20.31)$$

$$\mathbf{V}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1} \right) \cdot \left(\psi \vec{a}(s,t) \right) + \sum_{\vartheta=s+1}^{t} \left(\mathbf{B}(s,\vartheta) * \mathbf{1} \right) \cdot \left(\psi \vec{a}(s,\vartheta) \right) + \sum_{\vartheta=s+1}^{t} \mathbf{B}(\vartheta) * \left(\mathbf{V}(\vartheta,t)\nu(s,\vartheta) \right), \quad (20.32)$$

where respectively it holds:

$$\ddot{a}(t)=\sum_{h=1}^{t}\nu(h), \quad \ddot{a}(s,t)=\sum_{h=s+1}^{t}\nu(s,h).$$

20.3.1.3. *Third model*

The next step is the introduction of the variability of rewards so we assume that:

a) rewards are variable in time;

b) rewards are given only for the permanence in the state;

c) rewards are paid at the end of the period;

d) interest rate r is fixed.

In this case the following results hold:

$$\begin{aligned} V_i(t) &= (1-H_i(t))\sum_{h=1}^{t}\psi_i(h)\nu^h+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\sum_{h=1}^{\vartheta}\psi_i(h)\nu^h \\ &+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)V_k(t-\vartheta)\nu^{\vartheta}, \end{aligned} \tag{20.33}$$

$$\begin{aligned} V_i(s,t) &= (1-H_i(s,t))\sum_{h=s+1}^{t}\psi_i(h)\nu^{h-s}+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\sum_{h=s+1}^{\vartheta}\psi_i(h)\nu^{h-s} \\ &+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu^{\vartheta-s}. \end{aligned} \tag{20.34}$$

(20.33) and (20.34) in matrix form are:

$$\begin{aligned} \mathbf{V}(t) &= \left(\mathbf{D}(t)*\mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(t)*\underline{\nu}^{(t)}\right)+\sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta)*\mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(\vartheta)*\underline{\nu}^{(\vartheta)}\right) \\ &+\sum_{\vartheta=1}^{t}\mathbf{B}(\vartheta)*\left(\mathbf{V}(t-\vartheta)\nu^{\vartheta}\right), \end{aligned} \tag{20.35}$$

$$\begin{aligned} \mathbf{V}(s,t) &= \left(\mathbf{D}(s,t)*\mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(s,t)*\underline{\nu}^{(t-s)}\right) \\ &+\sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta)*\mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(s,\vartheta)*\underline{\nu}^{(\vartheta-s)}\right) \\ &+\sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta)*\left(\mathbf{V}(\vartheta,t)\nu^{\vartheta-s}\right), \end{aligned} \tag{20.36}$$

where

$$\underline{\boldsymbol{\psi}}(t)=\begin{bmatrix}\psi_1(1) & \psi_1(2) & \cdots & \psi_1(t)\\ \psi_2(1) & \psi_2(2) & \cdots & \psi_2(t)\\ \vdots & \vdots & \ddots & \vdots\\ \psi_m(1) & \psi_m(2) & \cdots & \psi_m(t)\end{bmatrix}, \underline{\nu}^{(h)}=\begin{bmatrix}\nu^1\\ \nu^2\\ \vdots\\ \nu^h\end{bmatrix}$$

and

$$\underline{\boldsymbol{\psi}}(s,t)=\begin{bmatrix}\psi_1(s+1) & \psi_1(s+2) & \cdots & \psi_1(t)\\ \psi_2(s+1) & \psi_2(s+2) & \cdots & \psi_2(t)\\ \vdots & \vdots & \ddots & \vdots\\ \psi_m(s+1) & \psi_m(s+2) & \cdots & \psi_m(t)\end{bmatrix}.$$

20.3.1.4. *Fourth model*

For the case of variable interest rates with variable rewards, we assume that:

a) rewards are variable in time;

b) rewards are given only for the permanence in the state;

c) rewards are paid at the end of the period;

d) interest rates are variable in time.

Here, the evolution equation takes the form:

$$\begin{aligned} V_i(t) \;=\; & (1-H_i(t))\sum_{h=1}^{t}\psi_i(h)\nu(h)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\sum_{h=1}^{\vartheta}\psi_i(h)\nu(h)\\ & +\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)V_k(t-\vartheta)\nu(\vartheta), \end{aligned} \tag{20.37}$$

$$V_i(s,t) \;=\; (1-H_i(s,t))\sum_{h=s+1}^{t}\psi_i(h)\nu(s,h)+$$

$$\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\sum_{h=s+1}^{\vartheta}\psi_i(h)\nu(s,h)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu(s,\vartheta). \tag{20.38}$$

Matrix forms of (20.37) and (20.38) respectively are:

$$\begin{aligned} \mathbf{V}(t) = & \left(\mathbf{D}(t)*\mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(t)*\underline{\nu}(t)\right)+\sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta)*\mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(\vartheta)*\underline{\nu}(\vartheta)\right)\\ & +\sum_{\vartheta=1}^{t}\mathbf{B}(\vartheta)*\left(\mathbf{V}(t-\vartheta)\nu(\vartheta)\right), \end{aligned} \tag{20.39}$$

$$\begin{aligned}\mathbf{V}(s,t) &= \left(\mathbf{D}(s,t) * \mathbf{1}\right) \cdot \left(\underline{\boldsymbol{\psi}}(s,t) * \underline{\nu}(s,t)\right) \\ &+ \sum_{\vartheta=s+1}^{t} \left(\mathbf{B}(s,\vartheta) * \mathbf{1}\right) \cdot \left(\underline{\boldsymbol{\psi}}(s,\vartheta) * \underline{\nu}(s,\vartheta)\right) \\ &+ \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) * \left(\mathbf{V}(\vartheta,t)\nu(s,\vartheta)\right)\end{aligned} \tag{20.40}$$

where

$$\underline{\nu}(t) = \begin{bmatrix} \nu(1) \\ \nu(2) \\ \vdots \\ \nu(t) \end{bmatrix}, \; \underline{\nu}(s,t) = \begin{bmatrix} \nu(s+1) \\ \nu(s+2) \\ \vdots \\ \nu(t) \end{bmatrix}.$$

20.3.1.5. *Fifth model*

The next step will introduce the γ rewards in the case of a fixed interest rate.

We have the following assumptions:

a) rewards are variable in time;

b) rewards are given for the permanence in the state and at a given transition;

c) rewards are paid at the end of the period;

d) interest rate r is fixed.

Under these hypotheses, the homogenous general formula is the following:

$$\begin{aligned}V_i(t) \; &= \; (1 - H_i(t))\sum_{h=1}^{t} \psi_i(h)\nu^h + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\sum_{h=1}^{\vartheta}\psi_i(h)\nu^h \\ &+ \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\gamma_{ik}(\vartheta)\nu^{\vartheta} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)V_k(t-\vartheta)\nu^{\vartheta}.\end{aligned} \tag{20.41}$$

Here too, the meaning of relation (20.41) can be easily understood with a subdivision into four parts.

Due to of the presence of lump sums in the RMPV, given or taken at change of state times, let us say that the sum of the last two terms

$$\sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\gamma_{ik}(\vartheta)\nu^{\vartheta} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)V_k(t-\vartheta)\nu^{\vartheta} \tag{20.42}$$

concerning the rewards $\gamma_{ik}(\vartheta)$ are paid or received at the transition moment and so must be discounted for a time of ϑ periods as $V_k(t-\vartheta)$.

The corresponding non-homogenous formula is the following:

$$\begin{aligned} V_i(s,t) \;=\; & (1-H_i(s,t))\sum_{h=s+1}^{t}\psi_i(h)\nu^{h-s} + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\sum_{h=s+1}^{\vartheta}\psi_i(h)\nu^{h-s} \\ & + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\gamma_{ik}(\vartheta)\nu^{\vartheta-s} + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu^{\vartheta-s}. \end{aligned} \tag{20.43}$$

Matrix forms of (20.41) and (20.43) are:

$$\begin{aligned} \mathbf{V}(t) = & \left(\mathbf{D}(t) * \mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(t) * \underline{\nu}^{(t)}\right) + \sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta) * \mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(\vartheta) * \underline{\nu}^{(\vartheta)}\right) \\ & + \sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta)\cdot\left(\boldsymbol{\Gamma}(\vartheta)\nu^{\vartheta}\right)\right) * \mathbf{1} + \sum_{\vartheta=1}^{t}\mathbf{B}(\vartheta) * \left(\mathbf{V}(t-\vartheta)\nu^{\vartheta}\right), \end{aligned} \tag{20.44}$$

$$\begin{aligned} \mathbf{V}(s,t) = & \left(\mathbf{D}(s,t) * \mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(s,t) * \underline{\nu}^{(t-s)}\right) + \sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta) * \left(\mathbf{V}(\vartheta,t)\nu^{\vartheta-s}\right) \\ & + \sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta)\cdot\left(\boldsymbol{\Gamma}(\vartheta)\nu^{\vartheta-s}\right)\right) * \mathbf{1} + \sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta) * \mathbf{1}\right)\cdot\left(\underline{\boldsymbol{\psi}}(s,\vartheta) * \underline{\nu}^{(\vartheta-s)}\right). \end{aligned} \tag{20.45}$$

20.3.1.6. *Sixth model*

The next model extends the preceding model with the variability of interest rates that is under the following assumptions:

a) rewards are variable in time;

b) rewards are given for the permanence in the state and at a given transition;

c) rewards are paid at the end of the period;

d) interest rates are variable in time.

All these hypotheses lead us to the following relations:

$$\begin{aligned} V_i(t) \;=\; & (1-H_i(t))\sum_{h=1}^{t}\psi_i(h)\nu(h) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\sum_{h=1}^{\vartheta}\psi_i(h)\nu(h) \\ & + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\gamma_{ik}(\vartheta)\nu(\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)V_k(t-\vartheta)\nu(\vartheta). \end{aligned} \tag{20.46}$$

$$V_i(s,t) = (1 - H_i(s,t)) \sum_{h=s+1}^{t} \psi_i(h)\nu(s,h) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \sum_{h=s+1}^{\vartheta} \psi_i(h)\nu(s,h)$$
$$+ \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\gamma_{ik}(\vartheta)\nu(s,\vartheta) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu(s,\vartheta). \tag{20.47}$$

(20.46) and (20.47) matrix forms are:

$$\mathbf{V}(t) = (\mathbf{D}(t) * \mathbf{1}) \cdot \left(\underline{\boldsymbol{\psi}}(t) * \underline{\nu}(t) \right) + \sum_{\vartheta=1}^{t} (\mathbf{B}(\vartheta) * \mathbf{1}) \cdot \left(\underline{\boldsymbol{\psi}}(\vartheta) * \underline{\nu}(\vartheta) \right)$$
$$+ \sum_{\vartheta=1}^{t} (\mathbf{B}(\vartheta) \cdot (\boldsymbol{\Gamma}(\vartheta)\nu(\vartheta))) * \mathbf{1} + \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) * (\mathbf{V}(t-\vartheta)\nu(\vartheta)), \tag{20.48}$$

$$\mathbf{V}(s,t) = (\mathbf{D}(s,t) * \mathbf{1}) \cdot \left(\underline{\boldsymbol{\psi}}(s,t) * \underline{\nu}(s,t) \right)$$
$$+ \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) * (\mathbf{V}(\vartheta,t)\nu(s,\vartheta)) + \sum_{\vartheta=s+1}^{t} (\mathbf{B}(s,\vartheta) \cdot (\boldsymbol{\Gamma}(\vartheta)\nu(s,\vartheta))) * \mathbf{1} \tag{20.49}$$
$$+ \sum_{\vartheta=s+1}^{t} (\mathbf{B}(s,\vartheta) * \mathbf{1}) \cdot \left(\underline{\boldsymbol{\psi}}(s,\vartheta) * \underline{\nu}(s,\vartheta) \right).$$

20.3.1.7. *Seventh model*

For our last case, we consider non-homogenous rewards and interest rate. Therefore, the basic assumptions are:

a) rewards are non-homogenous;

b) rewards are also given at the transitions;

c) rewards are paid at the end of the period;

d) interest rate is non-homogenous.

It can easily be verified that the evolution equation takes the form:

$$V_i(s,t) = (1 - H_i(s,t)) \sum_{\tau=s+1}^{t} \psi_i(s,\tau)\dot{\nu}(s,\tau)$$
$$+ \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \sum_{\tau=s+1}^{\vartheta} \psi_i(s,\tau)\dot{\nu}(s,\tau) \tag{20.50}$$
$$+ \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\dot{\nu}(s,\vartheta)\gamma_{ik}(s,\vartheta) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\dot{\nu}(s,\vartheta)V_k(\vartheta,t).$$

(20.50) in matrix form becomes

$$\begin{aligned}\mathbf{V}(s,t) &= \mathbf{D}(s,t)\sum_{\tau=s+1}^{t}\boldsymbol{\psi}(s,\tau)\dot{\nu}(s,\tau)\\&+\sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta)*\sum_{\tau=s+1}^{\vartheta}\boldsymbol{\psi}(s,\tau)\dot{\nu}(s,\tau)\\&+\sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta)\cdot\left(\boldsymbol{\Gamma}(s,\vartheta)\dot{\nu}(s,\vartheta)\right)\right)*\mathbf{1}+\sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta)*\left(\mathbf{V}(\vartheta,t)\dot{\nu}(s,\vartheta)\right)\end{aligned} \tag{20.51}$$

To conclude this section, we will present the most significant due cases. The reasoning is quite similar to the models for the immediate case but nevertheless, it is useful to classify the most interesting models.

As above, we systematically treat the homogenous and non-homogenous cases.

20.3.2. *The due cases*

20.3.2.1. *First model*

For the due case, our first model has the following assumptions

a) rewards are fixed in time;

b) rewards are given only for the permanence in the state;

c) rewards are paid at the beginning of the period;

d) interest rate *r* is fixed.

Here, $\ddot{V}_i(t)$ $(\ddot{V}_i(s,t))$ represents the RMPV given that at time 0, (*s*) the system in state *i* and the rewards being paid at the beginning of the period.

Under our hypotheses, the evolution equations take the form:

$$\ddot{V}_i(t) = (1-H_i(t))\psi_i\ddot{a}_{\overline{t}|r}+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\psi_i\ddot{a}_{\overline{\vartheta}|r}+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\ddot{V}_k(t-\vartheta)\nu^{\vartheta}, \tag{20.52}$$

$$\begin{aligned}\ddot{V}_i(s,t) &= (1-H_i(s,t))\psi_i\ddot{a}_{\overline{t-s}|r}+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\psi_i\ddot{a}_{\overline{\vartheta-s}|r}\\&\quad+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t)\nu^{\vartheta-s}.\end{aligned} \tag{20.53}$$

The matrix forms of (20.52) and (20.53) respectively are

$$\ddot{\mathbf{V}}(t) = \left(\mathbf{D}(t) * \mathbf{1} \right) \cdot \left(\boldsymbol{\psi} \ddot{a}_{\overline{t}|r} \right) + \sum_{\vartheta=1}^{t} \left(\mathbf{B}(\vartheta) * \mathbf{1} \right) \cdot \left(\boldsymbol{\psi} \ddot{a}_{\overline{\vartheta}|r} \right) + \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) * \left(\ddot{\mathbf{V}}(t-\vartheta) \nu^{\vartheta} \right) \tag{20.54}$$

$$\ddot{\mathbf{V}}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1} \right) \cdot \left(\boldsymbol{\psi} \ddot{a}_{\overline{t-s}|r} \right) + \sum_{\vartheta=s+1}^{t} \left(\mathbf{B}(s,\vartheta) * \mathbf{1} \right) \cdot \left(\boldsymbol{\psi} \ddot{a}_{\overline{\vartheta-s}|r} \right) + \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) * \left(\ddot{\mathbf{V}}(\vartheta,t) \nu^{\vartheta-s} \right) \tag{20.55}$$

20.3.2.2. *Second model*

We now consider variable rewards and variable interest rates to obtain the following assumptions:

a) rewards are variable in time,

b) rewards are given only for the permanence in the state,

c) rewards are paid at the beginning of the period,

d) interest rates are time dependent.

The related evolution equations are:

$$\ddot{V}_i(t) = (1 - H_i(t)) \sum_{\tau=0}^{t-1} \psi_i(\tau+1) \nu(\tau) + \sum_{k=1}^{m} \sum_{\vartheta=1}^{t} b_{ik}(\vartheta) \sum_{\tau=0}^{\vartheta-1} \psi_i(\tau+1) \nu(\tau) + \sum_{k=1}^{m} \sum_{\vartheta=1}^{t} b_{ik}(\vartheta) \ddot{V}_k(t-\vartheta) \nu(\vartheta), \tag{20.56}$$

$$\ddot{V}_i(s,t) = (1 - H_i(s,t)) \sum_{\tau=s}^{t-1} \psi_i(\tau+1) \nu(s,\tau) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \sum_{\tau=s}^{\vartheta-1} \psi_i(\tau+1) \nu(s,\tau) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \ddot{V}_k(\vartheta,t) \nu(s,\vartheta). \tag{20.57}$$

The matrix forms of (20.56) and (20.57) are

$$\ddot{\mathbf{V}}(t) = \left(\mathbf{D}(t) * \mathbf{1} \right) \cdot \left(\ddot{\underline{\boldsymbol{\psi}}}(t) * \underline{\ddot{\nu}}(t) \right) + \sum_{\vartheta=1}^{t} \left(\mathbf{B}(\vartheta) * \mathbf{1} \right) \cdot \left(\ddot{\underline{\boldsymbol{\psi}}}(\vartheta) * \underline{\ddot{\nu}}(\vartheta) \right) + \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) * \left(\ddot{\mathbf{V}}(t-\vartheta) \nu(\vartheta) \right) \tag{20.58}$$

$$\ddot{\mathbf{V}}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1}\right) \cdot \left(\underline{\ddot{\boldsymbol{\psi}}}(s,t) * \underline{\ddot{\nu}}(s,t)\right)$$
$$+ \sum_{\vartheta=s+1}^{t} \left(\mathbf{B}(s,\vartheta) * \mathbf{1}\right) \cdot \left(\underline{\ddot{\boldsymbol{\psi}}}(s,\vartheta) * \underline{\ddot{\nu}}(s,\vartheta)\right) + \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) * \left(\ddot{\mathbf{V}}(\vartheta,t)\nu(s,\vartheta)\right) \tag{20.59}$$

where

$$\underline{\ddot{\boldsymbol{\psi}}}(t) = \begin{bmatrix} \psi_1(1) & \psi_1(2) & \cdots & \psi_1(t) \\ \psi_2(1) & \psi_2(2) & \cdots & \psi_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_m(1) & \psi_m(2) & \cdots & \psi_m(t) \end{bmatrix}, \; \underline{\ddot{\nu}}(t) = \begin{bmatrix} 1 \\ \nu(1) \\ \vdots \\ \nu(t-1) \end{bmatrix}$$

and

$$\underline{\ddot{\boldsymbol{\psi}}}(s,t) = \begin{bmatrix} \psi_1(s+1) & \psi_1(s+2) & \cdots & \psi_1(t) \\ \psi_2(s+1) & \psi_2(s+2) & \cdots & \psi_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_m(s+1) & \psi_m(s+2) & \cdots & \psi_m(t) \end{bmatrix}, \; \underline{\ddot{\nu}}(s,t) = \begin{bmatrix} 1 \\ \nu(s,s+1) \\ \vdots \\ \nu(s,t-1) \end{bmatrix}.$$

20.3.2.3. *Third model*

With the introduction of γ rewards and with a fixed interest rate, the assumptions of our third model are:

a) rewards are variable in time;

b) rewards are given for the permanence in the state and at a given transition;

c) rewards are paid at the beginning of the period;

d) interest rate r is fixed.

Under these hypotheses the equations are:

$$\ddot{V}_i(t) = (1 - H_i(t))\sum_{\tau=0}^{t-1}\psi_i(\tau+1)\nu^{\tau} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\sum_{\tau=0}^{\vartheta-1}\psi_i(\tau+1)\nu^{\tau}$$
$$+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\nu^{\vartheta} b_{ik}(\vartheta)\ddot{V}_k(t-\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\nu^{\vartheta} b_{ik}(\vartheta)\gamma_{ik}(\vartheta), \tag{20.60}$$

$$\ddot{V}_i(s,t) = \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} \nu^{\vartheta-s} b_{ik}(s,\vartheta)\gamma_{ik}(\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\sum_{\tau=s}^{\vartheta-1}\psi_i(\tau+1)\nu^{\tau-s}$$
$$+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} \nu^{\vartheta-s} b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t) + (1-H_i(s,t))\sum_{\tau=s}^{t-1}\psi_i(\tau+1)\nu^{\tau-s}. \quad (20.61)$$

(20.60) and (20.61) matrix forms are:

$$\ddot{\mathbf{V}}(t) = \left(\mathbf{D}(t) * \mathbf{1}\right)\cdot\left(\underline{\ddot{\boldsymbol{\psi}}}(t) * \underline{\ddot{\nu}}^{(t)}\right) + \sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta) * \mathbf{1}\right)\cdot\left(\underline{\ddot{\boldsymbol{\psi}}}(\vartheta) * \underline{\ddot{\nu}}^{(\vartheta)}\right)$$
$$+\sum_{\vartheta=1}^{t}\left(\mathbf{B}(\vartheta)\cdot\left(\boldsymbol{\Gamma}(\vartheta)\nu^{\vartheta}\right)\right)*\mathbf{1} + \sum_{\vartheta=1}^{t}\mathbf{B}(\vartheta) * \left(\ddot{\mathbf{V}}(t-\vartheta)\nu^{\vartheta}\right) \quad (20.62)$$

$$\ddot{\mathbf{V}}(s,t) = \left(\mathbf{D}(s,t) * \mathbf{1}\right)\cdot\left(\underline{\ddot{\boldsymbol{\psi}}}(s,t) * \underline{\ddot{\nu}}^{(t-s)}\right) + \sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta) * \left(\ddot{\mathbf{V}}(\vartheta,t)\nu^{\vartheta-s}\right)$$
$$+\sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta)\cdot\left(\boldsymbol{\Gamma}(\vartheta)\nu^{\vartheta-s}\right)\right)*\mathbf{1} + \sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta) * \mathbf{1}\right)\cdot\left(\underline{\ddot{\boldsymbol{\psi}}}(s,\vartheta) * \underline{\ddot{\nu}}^{(\vartheta-s)}\right) \quad (20.63)$$

where

$$\underline{\ddot{\nu}}^{(t)} = \begin{bmatrix} \nu^0 \\ \nu^1 \\ \vdots \\ \nu^{t-1} \end{bmatrix} = \begin{bmatrix} 1 \\ (1+r)^{-1} \\ \vdots \\ (1+r)^{-t+1} \end{bmatrix}.$$

20.3.2.4. *Fourth model*

Our last model introduces non-homogenous rewards and interest rates with the following assumptions:

a) rewards are non-homogenous in time;

b) rewards are also given at the transitions;

c) rewards are paid at the beginning of the period;

d) the interest rate is non-homogenous.

For this, the evolution equation has the following form:

$$
\begin{aligned}
\ddot{V}_i(s,t) \;=\; & (1-H_i(s,t))\sum_{\tau=s}^{t-1}\psi_i(s,\tau+1)\dot{v}(s,\tau) \\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\sum_{\tau=0}^{\vartheta-1}\psi_i(s,\tau+1)\dot{v}(s,\tau) \\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\dot{v}(s,\vartheta)b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\dot{v}(s,\vartheta)b_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta).
\end{aligned}
\tag{20.64}
$$

The matrix form of (20.64) is given by

$$
\begin{aligned}
\ddot{\mathbf{V}}(s,t) = & \left(\mathbf{D}(s,t)\cdot\sum_{\tau=s}^{t-1}\ddot{\boldsymbol{\psi}}(s,\tau)\dot{v}(s,\tau)\right) * \mathbf{1} \\
&+\sum_{\vartheta=s+1}^{t}\mathbf{B}(s,\vartheta) * \left(\ddot{\mathbf{V}}(\vartheta,t)v(s,\vartheta)\right)+\sum_{\vartheta=s+1}^{t}\left(\mathbf{B}(s,\vartheta)\cdot\left(\boldsymbol{\Gamma}(s,\vartheta)\dot{v}(s,\vartheta)\right)\right) * \mathbf{1} \\
&+\sum_{\vartheta=s+1}^{t}\left(\left(\mathbf{B}(s,\vartheta)\cdot\sum_{\tau=s}^{\vartheta-1}\ddot{\boldsymbol{\psi}}(s,\tau)\dot{v}(s,\tau)\right) * \mathbf{1}\right)
\end{aligned}
\tag{20.65}
$$

20.4. MRWP and stochastic annuities

20.4.1. *Stochastic annuities*

The annuity concept is very simple and can easily be understood by means of the following figure.

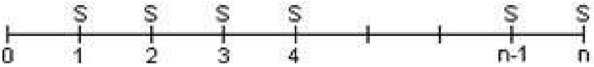

Figure 20.1. *Constant payment annuity-immediate*

where S represents the constant annuity payment.

Figure 20.1 shows the simplest immediate case.

The due case can be shown by the following figure.

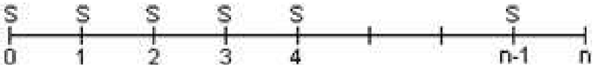

Figure 20.2. *Constant payment annuity-due*

Clearly, the payment can be variable. The simple problems to be dealt with are how to calculate the value at time 0 (present value) or at time *n* (capitalization value) of the annuity (see the first half of this book)

S can be considered not as a simple variable but rather as a random variable. This r.v. can assume, in the case of payments that vary only because of state, the following values:

$$S = \{ S_1, S_2, \ldots, S_m \}, \tag{20.66}$$

where S_i can be considered as the payment related to state *i*.

Furthermore, if it is set that the value at time *k* will depend only on the value at time *k*-1, we are in Markov process hypotheses. A sum is associated with each state which means that we are in a Markov reward environment. The problem of calculating the present value of this first simple case corresponds to the simplest case of DTHMRWP presented.

In this light, it now is possible to give the following definition.

Definition 20.2 *Discrete-time homogenous (non-homogenous) constant stochastic annuity*

Let:

$$I = \{1, 2, \ldots, m\}$$

be the states of a system and *A*, *B* two persons.

Furthermore, let

$$\{S_1, S_2, \ldots, S_m\},\ S_i \in \mathbb{R} \tag{20.67}$$

be sums.

The sum S_i will be paid or received from *A* to *B* if the system is in state *i*. These "payments" will be made from time $s+1$ [respectively *s*] up to time $s+n=T$ [respectively $s+n-1=T-1$].

We say that this financial operation is an *immediate* [respectively *due*] *homogenous* (*non-homogenous*) *discrete-time constant stochastic annuity* if:

i) the transitions among the states are governed by a homogenous (non-homogenous) discrete-time Markov Chain **P** $\left(\mathbf{P}(t) = \left[p_{ij}(t) \right]\right)$;

ii) when there is a transition from *i* to *j*, it is possible that a sum γ_{ij} is paid or received.

Each stochastic annuity can be seen as a discrete-time Markov reward process. The randomness is given by the fact that the periodic payment annuity is a r.v. Also, transition payments are allowed.

In the case of a simple immediate annuity, Figure 20.1 becomes Figure 20.3, and the annuity value can assume one of the values of r.v. (20.66).

Figure 20.4 gives the corresponding due case.

We are concerned with outlining the fact that, by means of the figures, it is possible to see quite easily that Markov reward processes can be considered a natural generalization of the annuity concept.

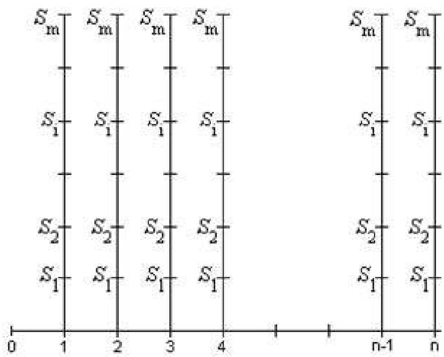

Figure 20.3. *Constant stochastic annuity-immediate*

It is clear that the reward structure could have a more complex structure that, in any case, can be seen as a generalization of the example shown in the two figures.

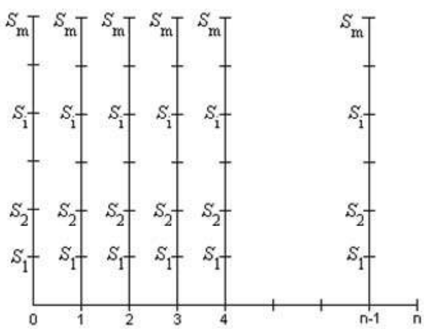

Figure 20.4. *Constant stochastic annuity-due*

It should be stated that this approach is not new in the actuarial environment; see, for example, Wolthuis (2003) and Daniel (2004). By means of our approach it is carried out in a more systematic way using the Markov reward process as the *natural stochastic generalization of the annuity* concept.

It is our opinion that the connection between Markov reward processes and annuities is natural and that an annuity can be seen as the Markov reward process with only one state and only permanence rewards.

In this light, within the field of finance it is possible to define *Markov reward processes* as *stochastic annuities.*

This first step also allows the generalization of the payments of the annuities in case of permanence rewards and transition rewards. Furthermore, the permanence rewards can be dependent or independent on the transition. All these rewards can be fixed or can vary due to time.

In the case of simple annuity, the payment can only vary due to time yet, in the case of *stochastic annuity,* clearly it can vary in the same way as the rewards, since rewards represent the payment generalization.

20.4.2. *Motorcar insurance application*

Stochastic annuities have many applications in the fields of finance and insurance.

In a general sense, actuarial mathematics can be seen as a branch of financial mathematics. In any actuarial mathematics application, we have to tackle a stochastic event within a financial environment. As it is well known, actuarial mathematics uses mathematical tools for insurance problems. In this light, DTMRWP could be seen as a useful tool to directly solve insurance problems.

In this section, DTMRWP will be applied to motor car *bonus malus* insurance rules that apply in Italy.

For a general reference on *bonus malus* systems and their properties, see Lemaire (1995) and Sundt (1993).

This example will use a transition matrix related to the motor car *bonus malus* insurance rules that apply in Italy. In this case, the Markov model fits quite well because:

1) the position of each insured person is given at the beginning of each year;

2) there are precise rules that give the change of states as a function of the behavior of the insured person during the year;

3) the future state depends only on the present one.

The number of states is 18.

Table 20.1 gives the evolution rules that hold in Italy for *bonus malus* insurance contract.

	Arriving state according to claims				
Starting state	0 claim	1 claim	2 claims	3 claims	4 or more
1	1	3	6	9	12
2	1	4	7	10	13
3	2	5	8	11	14
4	3	6	9	12	15
5	4	7	10	13	16
6	5	8	11	14	17
7	6	9	12	15	18
8	7	10	13	16	18
9	8	11	14	17	18
10	9	12	15	18	18
11	10	13	16	18	18
12	11	14	17	18	18
13	12	15	18	18	18
14	13	16	18	18	18
15	14	17	18	18	18
16	15	18	18	18	18
17	16	18	18	18	18
18	17	18	18	18	18

Table 20.1. *Italian bonus malus evolution rules*

We are in possession of the history of 105,627 insured persons over a period of three years (1998, 1999, 2000). This means that it was possible consider 316,881 real or virtual transitions. The Markov transition matrix that was obtained from the available data and taking into account the bonus malus Italian rules is given in Tables 20.2, 20.3 and 20.4.

States	1	2	3	4	5	6
1	0.941655	0	0.056264	0	0	0.001973
2	0.935097	0	0	0.062379	0	0
3	0	0.941646	0	0	0.056611	0
4	0	0	0.948892	0	0	0.049364
5	0	0	0	0.945231	0	0
6	0	0	0	0	0.949204	0
7	0	0	0	0	0	0.934685
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0

Table 20.2. *Transition matrix I*

States	7	8	9	10	11	12
1	0	0	0.000081	0	0	0.000027
2	0.002427	0	0	0.000097	0	0
3	0	0.001574	0	0	0.000169	0
4	0	0	0.001744	0	0	0
5	0.052354	0	0	0.002314	0	0
6	0	0.04908	0	0	0.00157	0
7	0	0	0.061856	0	0	0.00339
8	0.92227	0	0	0.073137	0	0
9	0	0.914103	0	0	0.082621	0
10	0	0	0.923854	0	0	0.071989
11	0	0	0	0.92933	0	0
12	0	0	0	0	0.930156	0
13	0	0	0	0	0	0.937854
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0

Table 20.3. *Transition matrix II*

States	13	14	15	16	17	18
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0.000067	0	0	0.000034	0	0
6	0	0.000146	0	0	0	0
7	0	0	0.000069	0	0	0
8	0.004246	0	0	0.00026	0	0.000087
9	0	0.003185	0	0	0	0.000091
10	0	0	0.003827	0	0	0.00033
11	0.066723	0	0	0.003696	0	0.000251
12	0	0.066697	0	0	0.002994	0.000153
13	0	0	0.059651	0	0	0.002495
14	0.920681	0	0	0.074704	0	0.004615
15	0	0.885204	0	0	0.107143	0.007653
16	0	0	0.777568	0	0	0.222432
17	0	0	0	0.876733	0	0.123267
18	0	0	0	0	0.888614	0.111386

Table 20.4. *Transition matrix III*

The payment of a claim by the insurance company can be seen as a lump sum (impulse or transition reward) paid by the insurer to the insured person. The model can be used to follow the financial evolution of a motor car insurance contract.

In Table 20.5, the premiums (which can be seen as permanence rewards) that are paid in Naples for a car of 2,300 c.c. are reported.

The example is constructed from the point of view of the insurance company and premiums are income for the company. It should be noted that these values correspond to the real premiums paid by an insured person in 2001 and officially given on the website of Assicurazioni Generali for that year.

States	Permanence rewards
1	1,037.5
2	1,099.75
3	1,162
4	1,224.25
5	1,286.5
6	1,369.5
7	1,452.5
8	1,535.5
9	1,618.5
10	1,701.5
11	1,826
12	1,950.5
13	2,075
14	2,386.25
15	2,697.5
16	3,112.5
17	3,631.25
18	4,150

Table 20.5. *Naples premiums*

In this case, permanence and impulse rewards should increase roughly in line with the inflation rate. In this light and with the aim of simplification, we suppose that the rewards are fixed in time. It is clear that the model can manage time variable premiums and benefits.

We suppose that we have a yearly fixed discount factor of 1/1.03. In the model, a stochastic interest rate could be easily introduced (see Janssen and Manca (2002)), but we do not think that this aspect is central in the presentation of our model.

Tables 20.6, 20.7 and 20.8 give the mean values of the expenses that the insurance company should pay for the claims made by the insured person.

More clearly stated, the element –7,772.51 represents the expenses that, on average, the company has to pay for the two accidents that an insured person who was in state 1 (lowest *bonus malus* class) had and which then took him to state 6.

These tables were constructed starting from the observed data in our possession.

From the point of view of the model, the elements of these three tables are transition rewards. More precisely, and as already mentioned, they can be seen as lump sums (impulse rewards) paid by the company at the time of the accident. In this case, being expenses for the company, they are negative.

States	1	2	3	4	5	6
1	0	0	–2,185.57	0	0	–7,772.51
2	0	0	0	–1,956.4	0	0
3	0	0	0	0	–2,188.25	0
4	0	0	0	0	0	–2,853.19
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0

Table 20.6. *Mean insurance payments I*

States	7	8	9	10	11	12
1	0	0	–3,240.77	0	0	–7,728.78
2	–3,196.16	0	0	–9,004.43	0	0
3	0	–2,846.52	0	0	–4,498.34	0
4	0	0	–2,920.39	0	0	0
5	–2,245.02	0	0	–3,945.44	0	0
6	0	–2,676.12	0	0	–3,076.05	0
7	0	0	–2,086.66	0	0	–3,391.18
8	0	0	0	–2,198.02	0	0
9	0	0	0	0	–2,017.77	0
10	0	0	0	0	0	–2,103.01
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0

Table 20.7. *Mean insurance payments II*

States	13	14	15	16	17	18
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	–3,240.77	0	0	–6,274.95	0	0
6	0	–6,703.61	0	0	0	0
7	0	0	–1,572.09	0	0	0
8	–4,027.26	0	0	–3,286.39	0	–3,629.14
9	0	–6,397.63	0	0	0	–3,687.5
10	0	0	–4,931.93	0	0	–5,165.44
11	–3,110.63	0	0	–4,710.94	0	–5,993.19
12	0	–3,048.69	0	0	–3,893.94	–1,1602.3
13	0	0	–2,613.27	0	0	–8,271.51
14	0	0	0	–3,564.01	0	–4,145.45
15	0	0	0	0	–2,468.23	–7,356.78
16	0	0	0	0	0	–2,883.68
17	0	0	0	0	0	–3,764.32
18	0	0	0	0	0	–2,578.55

Table 20.8. *Mean insurance payments III*

Tables 20.9, 20.10 and 20.11 report the present values of the mean total rewards that the company earns in 1 year, in 2 years and so on up to 20 years. Each column represents the starting state at time 0.

The permanence reward (insurance premium) increases as a function of the state and therefore the money earned by the company increases as a function of the starting state as well.

The results are interesting and show that, in this case, the company will earn a lot of money from this kind of insurance contract. The illustrated case is very particular. In Naples, the premiums are higher than in the other parts of Italy, the car is big and for this reason too the premiums are very high.

	Starting state					
Years	1	2	3	4	5	6
1	902.77	972.89	1,036.64	1,082.56	1,163.11	1,236.34
2	1,791.82	1,866.47	1,935.73	2,048.02	2,191.17	2,312.57
3	2,656.91	2,732.23	2,809.19	2,930.74	3,083.67	3,270.51
4	3,497.68	3,573.83	3,652.22	3,775.21	3,941.71	4,143.27
5	4,314.34	4,390.67	4,469.36	4,594.66	4,764.41	4,969.33
6	5,107.32	5,183.70	5,262.80	5,388.66	5,559.25	5,768.62
7	5,877.27	5,953.71	6,032.93	6,158.92	6,330.56	6,541.29
8	6,624.83	6,701.29	6,780.54	6,906.73	7,078.71	7,289.89
9	7,350.64	7,427.09	7,506.39	7,632.64	7,804.75	8,016.42
10	8,055.31	8,131.77	8,211.08	8,337.36	8,509.59	8,721.46
11	8,739.46	8,815.92	8,895.24	9,021.54	9,193.83	9,405.77
12	9,403.68	9,480.15	9,559.48	9,685.79	9,858.10	10,070.11
13	10,048.57	10,125.03	10,204.36	10,330.68	10,503.01	10,715.05
14	10,674.67	10,751.13	10,830.46	10,956.78	11,129.12	11,341.18
15	11,282.53	11,359.00	11,438.33	11,564.65	11,736.99	11,949.07
16	11,872.69	11,949.16	12,028.49	12,154.81	12,327.16	12,539.24
17	12,445.66	12,522.13	12,601.46	12,727.78	12,900.13	13,112.22
18	13,001.94	13,078.41	13,157.74	13,284.06	13,456.42	13,668.51
19	13,542.02	13,618.49	13,697.82	13,824.14	13,996.50	14,208.59
20	14,066.37	14,142.84	14,222.17	14,348.49	14,520.85	14,732.94

Table 20.9. *Present values of Naples mean total rewards I*

	Starting state					
Years	7	8	9	10	11	12
1	1,315.92	1,361.69	1,436.54	1,534.53	1,606.13	1,740.04
2	2,469.70	2,593.59	2,753.00	2,900.66	3,052.08	3,286.69
3	3,492.55	3,674.17	3,909.79	4,134.20	4,367.87	4,657.60
4	4,382.20	4,634.06	4,938.46	5,222.52	5,525.47	5,891.88
5	5,228.55	5,505.30	5,835.84	6,187.59	6,556.49	6,983.41
6	6,034.35	6,319.67	6,678.49	7,060.54	7,457.96	7,950.03
7	6,809.20	7,103.35	7,473.42	7,867.06	8,296.11	8,822.13
8	7,560.03	7,857.72	8,232.06	8,638.05	9,081.02	9,621.21
9	8,287.45	8,586.54	8,965.12	9,376.39	9,824.94	10,380.44
10	8,992.84	9,293.23	9,673.65	10,087.09	10,541.51	11,104.04
11	9,677.48	9,978.47	10,359.67	10,775.25	11,232.34	11,797.92
12	10,341.97	10,643.23	11,025.15	11,441.72	11,899.99	12,468.63
13	10,986.98	11,288.47	11,670.74	12,087.76	12,547.16	13,117.28
14	11,613.17	11,914.78	12,297.21	12,714.64	13,174.59	13,745.41
15	12,221.08	12,522.75	12,905.32	13,322.96	13,783.17	14,354.63
16	12,811.27	13,112.98	13,495.62	13,913.36	14,373.81	14,945.60
17	13,384.26	13,686.00	14,068.67	14,486.50	14,947.07	15,519.02
18	13,940.55	14,242.30	14,625.01	15,042.88	15,503.51	16,075.60
19	14,480.63	14,782.40	15,165.12	15,583.01	16,043.70	16,615.86
20	15,004.99	15,306.76	15,689.48	16,107.40	16,568.11	17,140.31

Table 20.10. *Present values of Naples mean total rewards II*

	Starting state					
Years	13	14	15	16	17	18
1	1,903.62	2,109.18	2,386.09	2,489.76	3,180.75	3,871.15
2	3,538.69	3,915.36	4,368.44	4,882.38	5,882.82	6,518.34
3	4,997.82	5,490.12	6,098.49	6,905.42	8,018.71	8,901.15
4	6,314.29	6,882.64	7,623.49	8,611.49	9,873.82	10,918.70
5	7,472.94	8,130.11	8,978.07	10,108.28	11,489.33	12,626.28
6	8,505.23	9,232.41	10,168.21	11,433.54	12,903.74	14,115.84
7	9,409.12	10,207.05	11,226.54	12,600.42	14,143.75	15,430.51
8	10,243.01	11,083.99	12,154.42	13,629.21	15,242.53	16,587.94
9	11,019.16	11,881.88	13,001.76	14,549.49	16,208.08	17,608.71
10	11,750.15	12,634.42	13,783.26	15,376.53	17,078.46	18,518.44
11	12,451.75	13,347.40	14,511.96	16,143.98	17,872.64	19,335.34
12	13,126.18	14,027.61	15,206.00	16,861.44	18,604.95	20,088.86
13	13,776.58	14,683.14	15,869.25	17,538.23	19,294.90	20,791.12
14	14,406.36	15,315.68	16,505.98	18,185.46	19,949.58	21,452.49
15	15,016.42	15,927.19	17,120.85	18,806.47	20,574.67	22,083.25
16	15,607.81	16,519.76	17,715.31	19,404.45	21,175.93	22,687.72
17	16,181.59	17,094.19	18,290.79	19,982.49	21,755.83	23,269.37
18	16,738.36	17,651.32	18,848.70	20,541.89	22,316.27	23,831.17
19	17,278.73	18,191.95	19,389.78	21,083.84	22,858.99	24,374.67
20	17,803.26	18,716.63	19,914.72	21,609.38	23,384.97	24,901.09

Table 20.11. *Present values of Naples mean total rewards III*

20.5. DTSMRWP and generalized stochastic annuities (GSA)

20.5.1. *Generalized stochastic annuities (GSA)*

The semi-Markov reward process is a generalization of the Markov reward process.

In discrete-time, the generalization of the SMRWP has the property that the waiting time before a transition is a r.v.

In the discrete-time Markov case, the transitions occur at each time step (the d.f that rules the transition is geometric). They can be *real* transitions, in the case where the system that goes all over the given system changes the state, or *virtual* in the case in which after the transition it remains in the same state. However, at each period there is a transition.

In the discrete-time Markov chain case, the time evolution of a trajectory can be described by means of Figure 20.5.

The time evolution of a trajectory of a discrete-time semi-Markov process is described in Figure 20.6.

The behavior of the two trajectories is not the same. The transition time in semi-Markov processes is random. A virtual transition case is shown in Figure 20.6.

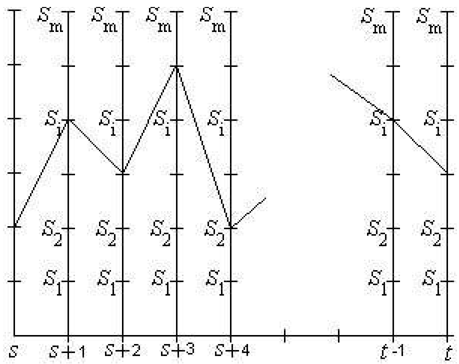

Figure 20.5. *A trajectory of discrete-time Markov reward process*

We attach a reward structure to the related process. In this light we can give the following definition.

Definition 20.3 A *generalized homogenous (non-homogenous) discrete-time stochastic annuity is a homogenous (non-homogenous) discrete-time stochastic annuity in which the following property holds:*

i) the transitions among the states follow a homogenous (non-homogenous) discrete-time semi-Markov process.

This financial concept naturally corresponds to the homogenous (non-homogenous) discrete-time semi-Markov reward process as defined in section 20.3.

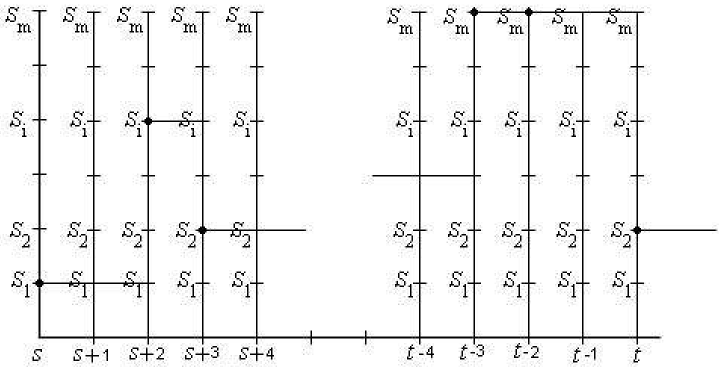

Figure 20.6. *A trajectory of DTNH semi-Markov reward process*

20.5.2. *GSA examples*

In Haberman and Pitacco (1999), Figure 20.7 is given to illustrate a trajectory of the stochastic process that describes a general insurance contract.

The depicted model has four states. It is evident that the transition time is naturally random.

A general insurance contract can be considered naturally to evolve in a semi-Markov environment. In the figures there are the states of the systems on the *y* axis, the time on the *x* axis; besides in Figure 20.7 the premiums and benefits of the insurance contract are also considered.

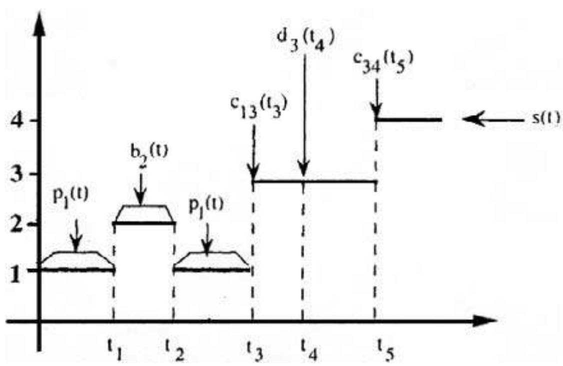

Figure 20.7. *Trajectory evolution of a general insurance contract*

Clearly, premiums and benefits can be considered as rewards. More precisely, in an HSMRWP environment we have the following:

• $p_1(t) = \psi_1(t)$ represents the premium paid by the insured. It is a permanence reward that can be constant or variable in time depending on the insurance contract;

• $b_2(t) = \psi_2(t)$ gives a benefit flow paid by the insurance company. Also in this case it is a constant or variable permanence reward;

• $d_3(t_4) = \psi_3(t)$ represents a discontinuous variable benefit flow where $\psi_3(t) = \begin{cases} k(t) & \text{if} \quad t \neq t_4 - t_3 \\ b & \text{if} \quad t = t_4 - t_3 \end{cases}$ clearly could also be $k(t) = 0 \ \ \forall t \neq t_4 - t_3$;

• $c_{13}(t_3) = \gamma_{13}(t)$ and $c_{34}(t_5) = \gamma_{34}(t)$ are transition rewards.

In this light we can say that any insurance contract can be modeled by means of SMRWP (MRWP can be seen as a particular case of SMRWP)!

In some cases, the homogenous environment is enough to model the insurance phenomenon. In other cases, the non-homogenity has to be introduced. Furthermore, in more composite cases the non-homogenous environment must be generalized to model the phenomenon (see Manca and Janssen (2007)).

In this first approach, we will consider the first examples that are reported in Haberman and Pitacco (1999). The related rewards evolution equations will be written.

The values that represent premiums and benefits have opposite algebraic signs. In these examples we will apply the discounted DTHSMRWP. Furthermore, we will suppose that the interest rate intensity δ is constant.

20.5.2.1. *Two states examples*

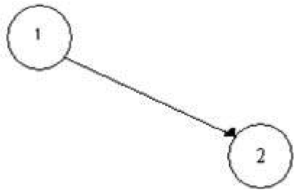

Figure 20.8. *1 = alive state; 2 = dead state*

Figure 20.8 can be used to depict three different cases of insurance:

(i) temporary assurance;

(ii) endowment assurance;

(iii) deferred annuity.

(i) In the case of death a constant sum *c* is assured and a premium at a constant rate *p* is paid at beginning of period. The policy ends at time *T*. So we have:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad t\{0,1\ldots,T-1\} \\ 0 & \text{if} \quad t \geq T \end{cases}, \qquad \gamma_{12} = c, \ 0 < t \leq T .$$

In all three cases, state 2 is an absorbing state and after time *T* the insurance contract is extinguished. The premiums are always anticipated. Furthermore, in this case $\psi_2 = 0$. $\psi_1(t)$ can be considered as a constant permanence reward and the evolution equations will follow the system for a time *T*.

The evolution equation is the equation with a fixed interest rate, fixed time due permanence rewards and fixed time transition reward.

Under all these hypotheses we can write the following evolution equations:

$$\begin{aligned} \ddot{V}_1(t) &= (1 - H_1(t))\ddot{a}_{\overline{t}|\delta}\psi_1 + \sum_{\vartheta=1}^{t} b_{12}(\vartheta)\ddot{a}_{\overline{\vartheta}|\delta}\psi_1 \\ &\quad + \sum_{\vartheta=1}^{t} b_{12}(\vartheta)\nu^{\vartheta}\gamma_{12} \end{aligned} \qquad 1 \leq t \leq T \tag{20.68}$$

$$\ddot{V}_2(t) = 0, \quad \forall t. \tag{20.69}$$

$\ddot{V}_1(t)$ represents the present value of the temporary assurance at time 0 for a time period *t* (backward reserve).

(ii) In the endowment assurance, a sum *c* is insured in both the cases of death and of survival to maturity *T*. We can have the following positions:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad t = \{1,\ldots,T\} \\ 0 & \text{if} \quad t > T \end{cases}, \quad \gamma_{12} = c, \ 0 < t \leq T$$

Also, in this case $\psi_2 = 0$. $\psi_1(t)$ can be considered as a variable permanence reward and the evolution equations will follow the system for a time *T*.

The evolution equation is the equation with a fixed interest rate, variable time permanence rewards and fixed time transition reward.

We can write the following evolution equation:

$$\ddot{V}_1(t) = (1 - H_1(t)) \sum_{\theta=0}^{t-1} \psi_1(\theta+1) v^{\theta} + \sum_{\vartheta=1}^{t} b_{12}(\vartheta) \sum_{\theta=0}^{\vartheta-1} \psi_1(\theta+1) v^{\theta} + \sum_{\vartheta=1}^{t} b_{12}(\vartheta) v^{\vartheta} \gamma_{12} \qquad 0 < t \le T$$

Also, in this case $\ddot{V}_1(t)$ represents the backward reserve at time 0 for a period *t* and (20.69) holds.

(iii) In the third case, the deferred annuity premiums are paid over the time period $\{1, \dots, T_1\}$ when the insured person is in state 1. Also, the benefits are paid continuously from time T_1 until the death of the insured, and as usual the premiums are anticipated and the claim amounts unknown, we recognize the well known *phenomenon of inversion of the production cycle* in insurance.

In this case we have:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad 1 \le t \le T_1, \\ b & \text{if} \quad T_1 \le t < \omega - x, \end{cases} \tag{20.70}$$

where ω represents the maximum age reachable by a person and *x* the insured age at the formation the contract.

In this case, $\psi_2 = 0$. $\psi_1(t)$ can be considered as a variable permanence reward and the evolution equations will follow the system for a time $\omega - x$.

The evolution equation is the equation with a fixed interest rate, variable time permanence rewards and no transition rewards.

We do not present this case but we can easily write the following evolution equation:

$$\ddot{V}_1(t) = (1 - H_1(t)) \sum_{\theta=0}^{t-1} \psi_1(\theta+1) v^{\theta} + \sum_{\vartheta=1}^{t} b_{12}(\vartheta) \sum_{\theta=0}^{\vartheta-1} \psi_1(\theta+1) v^{\theta} \tag{20.71}$$

In this case, $V_1(t)$ represents the backward reserve at time *t* and (20.69) holds.

In these three cases, the dead state does not give any permanence reward. It allows for the end of the contract and, in the first two cases, before the natural maturity.

Another two states example, given in Haberman and Pitacco, is as follows.

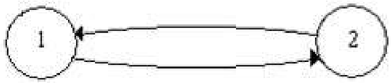

Figure 20.9. *1 = employed; 2 = unemployed*

In this case, the model can be used to study the annuity benefit in the case of unemployment. The dead state, in this two states model, is not considered because, as specified in Haberman and Pitacco (1999), the age range covered by such insurance contracts is characterized by low probabilities of death relative to the probabilities of moving from state 1 to state 2 or from state 2 to state 1, and because the financial effects of death may be small in relation to that of unemployment.

We will suppose that the premiums and benefits are fixed in time, but it is also possible to consider them variable without any difficulty. Under these hypotheses we obtain:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad 1 \le t \le T \\ 0 & \text{if} \quad t > T \end{cases} \quad \text{and} \quad \psi_2(t) = \begin{cases} b & \text{if} \quad 1 \le t \le T \\ 0 & \text{if} \quad t > T \end{cases} \tag{20.72}$$

where p is the premium paid by the insured, b is the benefit that he receives in the unemployment case and $T = W - x$, W is the maximum working age and x the insured age at the contractual formation. $\psi_1(t)$ and $\psi_2(t)$ could be considered as constant permanence rewards and the evolution equations will follow the system for a time T. In this case, because of the different period of payments we have a due case for premiums and an immediate case for claims.

The evolution equations will be the following:

$$\begin{aligned} \ddot{V}_1(t) = {} & (1 - H_1(t))\, \ddot{a}_{\overline{t|}\delta}\psi_1 + \sum_{\vartheta=1}^{t} b_{12}(\vartheta)\ddot{a}_{\overline{\vartheta|}\delta}\psi_1 \\ & + \sum_{\vartheta=1}^{t} b_{12}(\vartheta) v^{\vartheta} V_2(t-\vartheta) \end{aligned} \tag{20.73}$$

$$\begin{aligned} V_2(t) = {} & (1 - H_2(t))\, a_{\overline{t|}\delta}\psi_2 + \sum_{\vartheta=1}^{t} b_{21}(\vartheta) a_{\overline{\vartheta|}\delta}\psi_2 \\ & + \sum_{\vartheta=1}^{t} b_{12}(\vartheta) v^{\vartheta} \ddot{V}_1(t-\vartheta) \end{aligned} \tag{20.74}$$

(20.73) represents the mean present value that an insured has at time *t* if it starts at time 0 in state 1. (20.74) has the same meaning but this time starting in state 2.

In all the cases that we have considered there are no possibility of virtual transitions, which means that $p_{ii} = 0$ and so also $Q_{ii} = 0$ and in the evolution equation only b_{12} or b_{21} is considered.

20.5.2.2. *Three states examples*

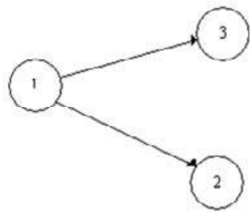

Figure 20.10. *Three state graph for the two three state examples*

Figure 20.10 can be used for the description of two cases:

(i) a temporary assurance with a rider benefit in the case of accidental death;

(ii) a lump sum benefit in the case of permanent and total disability.

In the first case, the three states will have the following meaning:

1 = alive;

2 = dead (other causes);

3 = dead (accident).

Two different causes of death are considered and the lump sums are a function of the death cause. We have:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad 1 \le t \le T \\ 0 & \text{if} \quad t > T \end{cases} \qquad \gamma_{12} = c, \ \ \gamma_{13} = c', \ \ 0 < t \le T$$

The evolution equation is similar to (20.68).

$$\ddot{V}_1(t) = \left(1 - H_1(t)\right) \ddot{a}_{\overline{t}|\delta} \psi_1 + \sum_{k=2}^{3} \sum_{\vartheta=1}^{t} b_{1k}(\vartheta) \ddot{a}_{\overline{\vartheta}|\delta} \psi_1 + \sum_{k=2}^{3} \sum_{\vartheta=1}^{t} b_{1k}(\vartheta) v^{\vartheta} \gamma_{1k} \qquad 0 < t \le T$$

$$V_2(t) = 0, \quad V_3(t) = 0, \quad \forall t. \tag{20.75}$$

In the other example related to Figure 20.10 a lump sum will be paid in the case of a permanent and total disability. The states are:

1 = active;

2 = disabled (permanent disability);

3 = dead.

The considered rewards are:

$$\psi_1(t) = \begin{cases} p & \text{if} \quad 0 < t \le T \\ 0 & \text{if} \quad t > T \end{cases} \qquad \gamma_{12} = c, \ 0 < t \le T$$

The evolution equation is the following:

$$\begin{aligned} \ddot{V}_1(t) &= (1 - H_1(t))\ddot{a}_{\overline{t|}\delta}\psi_1 + \sum_{k=2}^{3}\sum_{\vartheta=1}^{t} b_{1k}(\vartheta)\ddot{a}_{\overline{\vartheta|}\delta}\psi_1 \\ &\quad + \sum_{\vartheta=1}^{t} b_{12}(\vartheta)\nu^{\vartheta}\gamma_{12} \end{aligned} \tag{20.76}$$

(20.75) holds also in this case.

Remark 20.2 The time continuous version of the theory and other examples are given in Janssen and Manca (2006, 2007).